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Stability analysis of viscoelastic thin shallow hyperbolic paraboloid shells

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Abstract

The stability of linearly viscoelastic flexible shallow hyperbolic paraboloid shell is analysed under transverse load. Allowances are made for geometrical nonlinearity and initial imperfections of the surface shape. By application of the method of finite differences with respect to geometrical variables and the method of differentiation with respect to a parameter (time) the solution for the system of equilibrium non-linear integro-differential equations is reduced to Cauchy's problem which can be solved numerically. The critical time was shown to depend on the load, curvature, initial imperfections and edge elements compressibility. Critical loads for an outlying time moment are determined. © 1999 Elsevier Science Ltd. All rights reserved.

Nomenclature

a	half-span of shell (Fig. 1)
A_b	cross-sectional areas for edge element
D	bending stiffness ($= Eh^3/12(1-\nu^2)$)
E, ν	instantaneous elastic modulus and Poisson's ratio for shell
E_b	Young's modulus for edge element
f	half-rise of shell in the z-axis direction
h	shell thickness
k_{12}	curvature of middle surface ($=f/a^2$)
$\bar{k}_{12}$	$4k_{12}a^2/h$
K	membrane stiffness ($= Eh/(1-\nu^2)$)
M_x, M_y, M_{xy}	moments per unit of length
N_x, N_y, N_{xy}	membrane forces per unit of length
q	uniformly distributed transverse load
$\bar{q}$	$16qa^4/Eh^4$

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$\bar{q}_{cr}$	critical load parameter for an elastic shell
$\bar{q}_{cr}^{\infty}$	critical load parameter for a viscoelastic shell at an outlying time moment
$R(t-\tau)$	relaxation function for shell
t, τ	time ($\tau \leqslant t$)
t_{cr}	critical time
u, v, w	components of displacement at middle surfaces in the x-, y-, z-directions, respectively (positive in the co-ordinate axes direction)
$\bar{u}$, $\bar{v}$, $\bar{w}$	ua/h^2, va/h^2, w/h
w_0	deflection at shell centre
$\bar{w}_0$	w_0/h
w_i, w_{ia}	initial deflection and its amplitude
$\bar{w}_i$, $\bar{w}_{ia}$	w_i/h, w_{ia}/h
x, y, z	Cartesian co-ordinates
$()_{,x}$; $()_{,y}$ etc.	partial derivatives with respect to x or y, etc.

Greek symbols

α_1	$16(1+\nu)\ E_bA_b/Eha$
ε_b	longitudinal deformation in the edge element
ε_x, ε_y, ε_{xy}	middle surface strains
∇^4	biharmonic operator
ν_1, ν_2	$(1-\nu)/2$, $(1+\nu)/2$

1. Introduction

Shallow hyperbolic paraboloid shells (hypars) are used in various engineering fields, for example in civil engineering. The results of geometric non-linear stability analysis of an elastic flexible shallow hyperbolic paraboloid shell have been reported by Ishakov (1993). The present paper is dedicated to the problem of geometric non-linear stability in creep of the same type of shell made of linearly viscoelastic material. The problem of stability of viscoelastic shells has long been of interest to researchers. In Arutynyan et al. (1987), Kovarik (1987) they presented excellent reviews of studies concerning this problem. However, there are only a few researchers, which are lead up to numerical result. Among the published works we could not find any papers treating hypar-shells specifically. In this work prominence is given to the analysis of behaviour of a flexible shallow viscoelastic hypar-shell during the finite time-interval including the half-infinite interval. The critical time is determined according to Hoff's creep buckling criterion (see Hoff, 1956), i.e. on the assumption that the velocity of the shell deflection becomes infinitely large. We have considered the relationship between the critical time and load, surface curvature and initial imperfection of the shell shape, and determined the boundaries of stability for an outlying time moment.

First, the problem is formulated as the system of three non-linear integro-differential equations of equilibrium in terms of displacements. Then, by means of finite difference approximation with respect to geometrical variables it is reduced to the system of non-linear Volterra integral equations in nodal displacements. To provide a numerical solution to the system of non-linear integral equations we make use of the method of differentiation with respect to a parameter (time) which reduces the solution to the Cauchy's problem. This method was first proposed by Davidenko (1953) for solving the systems of non-linear equations. Later on it was applied in a number of works (mainly in the USSR), including the one mentioned above (Ishakov, 1993) to solution of geometrically non-linear problems pertaining to the theory of elastic plates and shells. This method favourably differs from other methods due to simplicity of implementation and possibility easily to pass through limit points of equilibrium path by change of the driving parameter (see e.g. Matevosyan, 1974; Ishakov, 1993). In the present paper, this method is extended to the solution of geometrically non-linear problems of stability of viscoelastic shell.

2. Statement of the problem and governing equations

Consider a flexible shallow hyperbolic paraboloid shell in the quadratic plane with the sizes of sides represented by $2a$ (Fig. 1). The origin of the co-ordinates coincides with the plane centre. In the system of co-ordinates x, y, z the ideal middle surface of the shell is described by the equation $z = k_{12}xy$. The surface curvature k_{12} is constant at all points.

The shell is made of a linearly viscoelastic material with bounded creep that is damped in time. Among such materials are concrete, certain types of polymers, etc. The material of the shell is assumed to be non-ageing. It means that its instantaneous elastic modulus is independent of time, i.e. it remains constant. Following Arutynyan (1952) (see also Olszak and Sawczuk, 1967) it is also assumed that the coefficients of transverse elastic deformation and transverse creep deformation are equal to each other and constant in time, i.e. $v(t) = v_c(t, \tau) = v = \text{const}$. This assumption is used often with a research of thin viscoelastic shells (see e.g. Deak, 1972), as a change of transverse deformation coefficient influences behaviour of a shell a little. The material of the shell is linearly elastic under instantaneous loading.

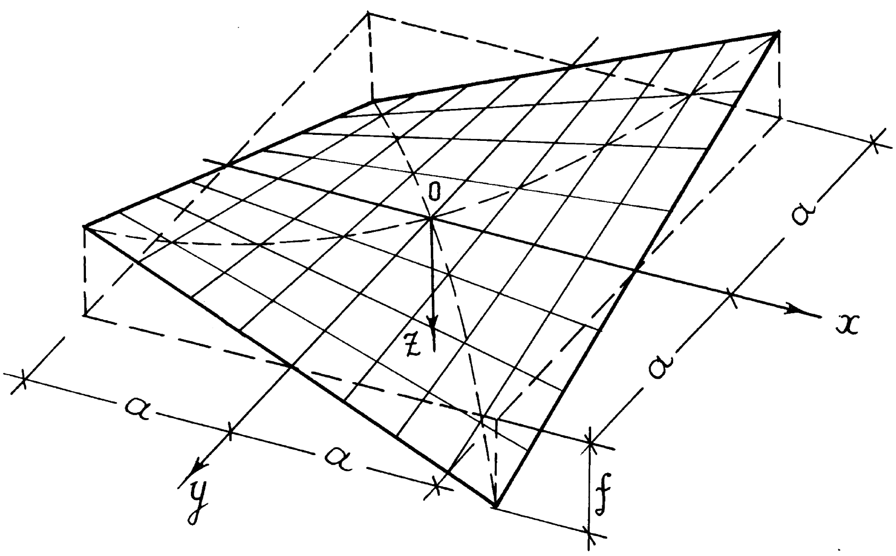

Fig. 1. Hypar-shell geometry.

At the rectilinear boundaries the shell is strengthened with elastic edge elements having constant extensional stiffness E_bA_b. In other respects, the shell is considered to be simply supported at the boundary. It is assumed that the shell edges can curve freely in the boundary plane and turn normally to the boundary (i.e. the horizontal bending and torsional stiffnesses of the edge elements are equal to zero). Deflections on the contour are equal to zero. The lower corners of the shell remain undisplaced.

The shell is subjected to the uniformly distributed transverse load q. The load is considered positive if it acts in the z-axis direction. The law of time loading is taken as a single-stage one. It means that the time count starts from the moment $t = 0$, when the shell is suddenly subjected to a distributed load of q intensity, which is then maintained at a constant level. With $t < 0$ the shell is considered non-loaded. The problem is solved in a quasi-static setting, i.e., without regard for inertia forces. Consideration is given to symmetrical forms of equilibrium.

In the present paper, to obtain governing equations we follow the classical scheme based on Kirchhoff–Love hypotheses. We also take into account the assumptions of the shallow shell theory, geometrical non-linearity and presence of geometrical initial imperfections with initial deflections w_{i}. The components of middle surface strains are expressed through components of displacements according to Karman–Marguerre geometrically non-linear theory (see e.g. Timoshenko and Woinowsky-Krieger, 1959; Donnell, 1976). The stress–strain dependences for the shell are given in Volterra integral form of linearly viscoelastic theory (see e.g. Bland, 1960). The system of three non-linear integro-differential equations of equilibrium in terms of displacements for a viscoelastic flexible shallow hypar-shell with initial deviations w_{i} from the ideal shape is expressed as follows:

$$
\begin{aligned}
&u(t)_{,xx}+v_1u(t)_{,yy}+v_2v(t)_{,xy}-2v_1k_{12}w(t)_{,y}+w(t)_{,x}[w(t)_{,xx}+w_{\mathrm{i},xx}]\\
&+v_1w(t)_{,x}[w(t)_{,yy}+w_{\mathrm{i},yy}]+v_2w(t)_{,y}[w(t)_{,xy}+w_{\mathrm{i},xy}]+[w(t)_{,xx}+v_1w(t)_{,yy}]w_{\mathrm{i},x}+v_2w(t)_{,xy}w_{\mathrm{i},y}\\
&=\int_0^t\{u(\tau)_{,xx}+v_1u(\tau)_{,yy}+v_2v(\tau)_{,xy}-2v_1k_{12}w(\tau)_{,y}+w(\tau)_{,x}[w(\tau)_{,xx}+w_{\mathrm{i},xx}]\\
&+v_1w(\tau)_{,x}[w(\tau)_{,yy}+w_{\mathrm{i},yy}]+v_2w(\tau)_{,y}[w(\tau)_{,xy}+w_{\mathrm{i},xy}]+[w(\tau)_{,xx}\\
&+v_1w(\tau)_{,yy}]w_{\mathrm{i},x}+v_2w(\tau)_{,xy}w_{\mathrm{i},y}\}R(t-\tau)\,\mathrm{d}\tau;\\
&v(t)_{,yy}+v_1v(t)_{,xx}+v_2u(t)_{,xy}-2v_1k_{12}w(t)_{,x}+w(t)_{,y}[w(t)_{,yy}+w_{\mathrm{i},yy}]\\
&+v_1w(t)_{,y}[w(t)_{,xx}+w_{\mathrm{i},xx}]+v_2w(t)_{,x}[w(t)_{,xy}+w_{\mathrm{i},xy}]+[w(t)_{,yy}+v_1w(t)_{,xx}]w_{\mathrm{i},y}+v_2w(t)_{,xy}w_{\mathrm{i},x}\\
&=\int_0^t\{v(\tau)_{,yy}+v_1v(\tau)_{,xx}+v_2u(\tau)_{,xy}-2v_1k_{12}w(\tau)_{,x}+w(\tau)_{,y}[w(\tau)_{,yy}+w_{\mathrm{i},yy}]\\
&+v_1w(\tau)_{,y}[w(\tau)_{,xx}+w_{\mathrm{i},xx}]+v_2w(\tau)_{,x}[w(\tau)_{,xy}+w_{\mathrm{i},xy}]+[w(\tau)_{,yy}\\
&+v_1w(\tau)_{,xx}]w_{\mathrm{i},y}+v_2w(\tau)_{,xy}w_{\mathrm{i},x}\}R(t-\tau)\,\mathrm{d}\tau; \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
&D\nabla^4 w(t) - K\Bigg\{[w(t)_{,xx} + w_{\mathrm{i},xx}]\Big\langle\Big[u(t)_{,x} + \frac{1}{2}(w(t)_{,x})^2 + w(t)_{,x}w_{\mathrm{i},x}\Big] + v\Big[v(t)_{,y} \\
&+ \frac{1}{2}(w(t)_{,y})^2 + w(t)_{,y}w_{\mathrm{i},y}\Big]\Big\rangle + [w(t)_{,yy} + w_{\mathrm{i},yy}]\Big\langle\Big[v(t)_{,y} + \frac{1}{2}(w(t)_{,y})^2 + w(t)_{,y}w_{\mathrm{i},y}\Big] \\
&+ v\Big[u(t)_{,x} + \frac{1}{2}(w(t)_{,x})^2 + w(t)_{,x}w_{\mathrm{i},x}\Big]\Big\rangle + 2v_1[k_{12} + w(t)_{,xy} + w_{\mathrm{i},xy}]\langle u(t)_{,y} + v(t)_{,x} \\
&- 2k_{12}w(t) + w(t)_{,x}w(t)_{,y} + w(t)_{,x}w_{\mathrm{i},y} + w(t)_{,y}w_{\mathrm{i},x}\rangle\Bigg\} - q \\
&= D\int_0^t \nabla^4 w(\tau)R(t-\tau)\,\mathrm{d}\tau - K\Bigg\{[w(t)_{,xx} + w_{\mathrm{i},xx}]\int_0^t\Big\langle\Big[u(\tau)_{,x} + \frac{1}{2}(w(\tau)_{,x})^2 \\
&+ w(\tau)_{,x}w_{\mathrm{i},x}\Big] + v\Big[v(\tau)_{,y} + \frac{1}{2}(w(\tau)_{,y})^2 + w(\tau)_{,y}w_{\mathrm{i},y}\Big]\Big\rangle R(t-\tau)\,\mathrm{d}\tau \\
&+ [w(t)_{,yy} + w_{\mathrm{i},yy}]\int_0^t\Big\langle\Big[v(\tau)_{,y} + \frac{1}{2}(w(\tau)_{,y})^2 + w(\tau)_{,y}w_{\mathrm{i},y}\Big] + v\Big[u(\tau)_{,x} + \frac{1}{2}(w(\tau)_{,x})^2 \\
&+ w(\tau)_{,x}w_{\mathrm{i},x}\Big]\Big\rangle R(t-\tau)\,\mathrm{d}\tau + 2v_1[k_{12} + w(t)_{,xy} + w_{\mathrm{i},xy}]\int_0^t\langle u(\tau)_{,y} + v(\tau)_{,x} \\
&- 2k_{12}w(\tau) + w(\tau)_{,x}w(\tau)_{,y} + w(\tau)_{,x}w_{\mathrm{i},y} + w(\tau)_{,y}w_{\mathrm{i},x}\rangle R(t-\tau)\,\mathrm{d}\tau\Bigg\}.
\end{aligned}
$$

All the displacements, deformations, internal forces and moments are functions of geometrical co-ordinates x, y and time t. Here in eqns (1) and further on, to make the expression shorter, they are given as functions of time t (or τ) only, dependence on co-ordinates x and y being implicit. Initial deflection w_{i} is dependent on co-ordinates x and y, and is not a function of time. In eqns (1) the relaxation function $R(t-\tau)$ is dependent on difference $t-\tau$ only, as the material is assumed to be non-ageing. For materials with bounded, damped creep the main requirement to be met by the relaxation function is that its integral, within the limits $(0, t)$ should have a finite value, even with $t \to \infty$. Otherwise the relaxation function can be of any experimentally confirmed form. For the moment of application of external load $t = 0$ the integral members in eqns (1) turn into zero. In this case the system of non-linear integro-differential eqns (1) turns into a system of non-linear differential equations of equilibrium in displacements for the elastic-instantaneous problem (Ishakov, 1993).

According to the shell supporting conditions on the contour described above the boundary conditions at the rectilinear edges can be given in the following form:

$$\text{at } x = \pm a: \quad w(t) = 0; \quad M_x(t) = 0; \quad N_x(t) = 0; \quad \varepsilon_y(t) = \varepsilon_b(t); \tag{2a}$$

$$\text{at } y = \pm a: \quad w(t) = 0; \quad M_y(t) = 0; \quad N_y(t) = 0; \quad \varepsilon_x(t) = \varepsilon_b(t). \tag{2b}$$

The fourth couple of boundary conditions (2a, b) express the joint deformation conditions of the viscoelastic shell and its elastic edge elements. The boundary conditions (2a, b) in terms of displacements are expressed as follows:

at $x = \pm a$:

$$w(t) = 0; \quad w(t)_{,xx} = 0; \quad u(t)_{,x} + \nu v(t)_{,y} + \frac{1}{2}(w(t)_{,x})^2 + w(t)_{,x} w_{i,x} = 0;$$

$$v(t)_{,yy} = \pm K\nu_1 \left\{ u(t)_{,y} + v(t)_{,x} - \int_0^t [u(\tau)_{,y} + v(\tau)_{,x}] R(t-\tau)\,\mathrm{d}\tau \right\} \Big/ (E_b A_b); \tag{3a}$$

at $y = \pm a$:

$$w(t) = 0; \quad w(t)_{,yy} = 0; \quad v(t)_{,y} + \nu u(t)_{,x} + \frac{1}{2}(w(t)_{,y})^2 + w(t)_{,y} w_{i,y} = 0;$$

$$u(t)_{,xx} = \pm K\nu_1 \left\{ u(t)_{,y} + v(t)_{,x} - \int_0^t [u(\tau)_{,y} + v(\tau)_{,x}] R(t-\tau)\,\mathrm{d}\tau \right\} \Big/ (E_b A_b). \tag{3b}$$

The availability of integral member in the boundary conditions causes redistribution of stresses in time owing to creep. At $t = 0$ the above written boundary conditions (3a) and (3b) on the sides $x = \pm a$ and $y = \pm a$ coincide with respective boundary conditions of the elastic-instantaneous problem (Ishakov, 1993).

Similar conditions at the corners are:

upper corners $x = a(-a)$, $y = -a(a)$:

$$w(t) = 0; \quad u(t)_{,x} = v(t)_{,y} = w(t)_{,x} = w(t)_{,y} = 0;$$

lower corners $x = y = a(-a)$:

$$u(t) = v(t) = w(t) = 0; \quad w(t)_{,x} = w(t)_{,y} = 0.$$

Conditions $u(t)_{,y} = v(t)_{,x} = 0$ resulting from the edge element continuity must be met at all the corners.

3. Method of solution

For numerical solution of the problem the non-linear integro-differential eqns (1), boundary (3) and corner conditions can be written in finite differences with respect to geometrical variables x, y with the order of error $O(\lambda^2)$. The difference interval is $\lambda = 2a/8$. The previous calculations (Ishakov, 1993) have shown that such grid (8×8) provides a sufficiently accurate solution. Further refinement of the mesh makes the calculation too complicated without significant improvement of its accuracy. Derivatives with respect to variables x, y are approximated by central differences. One-sided differences are used only for the corner points and for approximation of derivatives of initial deflection functions in the equations for the points on the contour and in the boundary conditions. Since this study covers only symmetrical forms of equilibrium, the difference equations are written only for nodes of the gird on a quarter of the shell field between diagonals $x = y$ and $x = -y$ (Fig. 1).

After substituting the finite differences for the derivatives with respect to variables x and y in eqns (1) we obtain, for the grid nodes under consideration, a system of non-linear Volterra integral equations in which nodal displacements u, v, w as functions of time are unknowns. The boundary and corner conditions are satisfied here in a conventional way. With introduction of dimensionless values $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{w}_{\mathrm{i}}$, $\bar{k}_{12}$, $\bar{q}$ the integral equations are transformed to the dimensionless form. The system of integral equations incorporates 55 equations, 39 of them being obtained from the first and the second eqns (1) and 16 from the third one. The quantity of unknown nodal displacements is equal to 55, respectively.

Let us denote the dimensionless nodal displacement values as x_k $(k = 1, 2, \ldots, 55)$: x_1, $x_2, \ldots$, x_{39}—tangential displacements $\bar{u}$, $\bar{v}$; $x_{40}, x_{41}, \ldots, x_{55}$—normal displacements $\bar{w}$. In actual and previous states all the variables x_k are unknown functions of time t or τ, respectively. Thus, in actual state they are: $x_1 = x_1(t)$, $x_2 = x_2(t), \ldots, x_{55} = x_{55}(t)$. The domain of these functions is $0 \leqslant \tau$, $t < t_{\mathrm{cr}}$.

In an implicit form, the system of non-linear integral equations can be written as:

$$F_i[x_1(t), x_2(t), \ldots, x_{55}(t); \bar{q}] = P_i(t), \quad (i = 1, 2, \ldots, 55). \tag{4}$$

Left-hand sides F_i of integral eqns (4) corresponding to those of eqns (1) are algebraic polynomials, up to the third degree inclusive, in dimensionless nodal displacements. Their right-hand sides P_i incorporating integrals with respect to time can be written in a generalized form as:

$$P_i^{(1)}(t) = \int_0^t f_{i1}(\tau) R(t-\tau)\,\mathrm{d}\tau, \quad (i = 1, 2, \ldots, 39);$$

$$P_i^{(2)}(t) = \int_0^t f_{i2}(\tau) R(t-\tau)\,\mathrm{d}\tau - 1{,}5 \sum_{j=3}^{5} \phi_{ij}(t) \int_0^t f_{ij}(\tau) R(t-\tau)\,\mathrm{d}\tau, \quad (i = 40, 41, \ldots, 55). \tag{5}$$

Here i = number of integral equation.

Expressions $P_i^{(1)}(t)$ are obtained from the right-hand side of the first and second equations of system (1) and expressions $P_i^{(2)}(t)$ are from the third one. Functions f_{i1}, f_{i2}, f_{ij}, ϕ_{ij} are algebraic polynomials in dimensionless nodal displacements:

$$f_{ij}(\tau) = f_{ij}[x_1(\tau), x_2(\tau), \ldots, x_{55}(\tau)], \quad (j = 1; 3, 4, 5);$$

$$f_{i2}(\tau) = f_{i2}[x_{40}(\tau), x_{41}(\tau), \ldots, x_{55}(\tau)];$$

$$\phi_{ij}(t) = \phi_{ij}[x_{40}(t), x_{41}(t), \ldots, x_{55}(t)], \quad (j = 3, 4, 5). \tag{6}$$

The functions $\phi_{ij}(t)$ are expressed by middle surface curvatures of actual state. The right-hand sides P_i of equations in system (4) are continuous functions of time t in domain $0 \leqslant t < t_{\mathrm{cr}}$. In eqns (4) load $\bar{q}$ is a constant with a given value.

Let us assume that

$$\text{at } t = 0: \quad x_1 = x_1^{(0)}, x_2 = x_2^{(0)}, \ldots, x_{55} = x_{55}^{(0)}. \tag{7}$$

Conditions (7) are the initial conditions of the shell creep problem.

To solve the system of integral eqns (4) at $t > 0$ we pass to Cauchy's problem. Here, we employ the method of differentiation with respect to a parameter (see Davidenko, 1953) taking time t as a driving parameter. Since the left-hand sides of eqns (4) are algebraic polynomials, all the functions $F_i (i = 1, 2, \ldots, 55)$ are continuous throughout the range of arguments $x_k (k = 1, 2, \ldots, 55)$ and have continuous partial derivatives of the first-order with respect to all the arguments. Taking time t as an independent variable we differentiate eqns (4) with respect to this variable. As a result we get a system of linear equations for the unknown velocities of dimensionless nodal displacements $\mathrm{d}x_k/\mathrm{d}t$ in actual state:

$$\sum_{k=1}^{55} (\partial F_i/\partial x_k)(\mathrm{d}x_k/\mathrm{d}t) = \mathrm{d}P_i/\mathrm{d}t, \quad (i = 1, 2, \ldots, 55) \tag{8}$$

or in the matrix form:

$$A(\mathrm{d}X/\mathrm{d}t) = B_t, \tag{9}$$

where the $\mathbf{A}$ (55×55) matrix and the $\mathbf{B}_t$ and $\mathbf{X}$ (55×1) vectors appear to be:

$$\mathbf{A} = \begin{bmatrix} \partial F_1/\partial x_1 & \partial F_1/\partial x_2 & \ldots & \partial F_1/\partial x_{55} \\ \partial F_2/\partial x_1 & \partial F_2/\partial x_2 & \ldots & \partial F_2/\partial x_{55} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \partial F_{55}/\partial x_1 & \partial F_{55}/\partial x_2 & \ldots & \partial F_{55}/\partial x_{55} \end{bmatrix}; \quad \mathbf{B}_t = \begin{bmatrix} \mathrm{d}P_1/\mathrm{d}t \\ \mathrm{d}P_2/\mathrm{d}t \\ \vdots \\ \vdots \\ \mathrm{d}P_{55}/\mathrm{d}t \end{bmatrix}; \quad \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{55} \end{bmatrix} \tag{10}$$

Let us assume that matrix $\mathbf{A}$ is non-singular at all the points of the domain. Then, solving system (9) we obtain:

$$\mathrm{d}\mathbf{X}/\mathrm{d}t = \mathbf{A}^{-1}\mathbf{B}_t \tag{11}$$

Vector $\mathbf{B}_t$ determines the contribution to the right-hand side of system (11) from preceding deformation history. To calculate its components it is necessary to keep in the data base the information about displacements obtained throughout the length of time $0-t$.

Initial conditions (7) can be written in the matrix form as:

$$\text{at } t = 0: \quad \mathbf{X} = \mathbf{X}_0, \tag{12}$$

where $\mathbf{X}_0 = [x_1^{(0)}, x_2^{(0)}, \ldots, x_{55}^{(0)}]$ is a vector of initial values of nodal displacements in the shell creep problem which can be found from the solution of elastic-instantaneous problem for the same shell subjected to the load of a given value. At $t = 0$ we have $P_i = 0$. Then the system of non-linear integral eqns (4) becomes a system of non-linear algebraic difference equations for elastic dimensionless nodal displacements $x_k (k = 1, 2, \ldots, 55)$:

$$F_i(x_1, x_2, \ldots, x_{55}, \bar{q}) = 0, \quad (i = 1, 2, \ldots, 55). \tag{13}$$

In this system the load parameter $\bar{q}$ is an independent variable running from zero to a given value, and variables x_k are unknown functions of load $\bar{q}$, i.e. $x_k = x_k(\bar{q})$. The system of eqns (13) obtained in this way together with its zero initial conditions

$$\text{at } \bar{q} = 0: \quad x_1 = x_2 = , \ldots, = x_{55} = 0, \tag{14}$$

provides a description for the elastic-instantaneous behaviour of the shell. It can be solved by the method of differentiation with respect to driving parameter $\bar{q}$ (or $\bar{w}_0$) as shown by Ishakov (1993). As a result, we obtain a system of linear equations relative to the unknowns $\mathrm{d}x_k/\mathrm{d}\bar{q}$, which in the matrix form can be expressed as follows:

$$\mathrm{d}\mathbf{X}/\mathrm{d}\bar{q} = \mathbf{A}^{-1}\mathbf{B}_q, \tag{15}$$

where $\mathbf{A}$ and $\mathbf{X}$ are matrix and vector (10),

$\mathbf{B}_q = [-\partial F_1/\partial\bar{q}, -\partial F_2/\partial\bar{q}, \ldots, -\partial F_{55}/\partial\bar{q}]$ is vector (55×1).

Initial conditions (14) are given as:

$$\text{at } \bar{q} = 0: \quad \mathbf{X} = 0 \tag{16}$$

To determine unknowns $x_k^{(0)} (k = 1,2,\ldots,55)$ represented by the $\mathbf{X}_0$ vector (12), the system of differential eqns (15) at initial conditions (16) is integrated numerically over the load parameter interval $0 \leqslant \bar{q} \leqslant$ 'given value' by means of step-by-step computation (by Euler/Runge–Kutta methods). For details, interested readers may refer to Davidenko (1953), Matevosyan (1974), Ishakov (1993).

A step-by-step computation technique with respect to time can be used to obtain a numerical solution of the system of eqns (11) under initial conditions (12). Thus, applying Euler's method we have the solution in the form:

$$\mathbf{X}_{n+1} = \mathbf{X}_n + \Delta t \mathbf{A}_n^{-1}\mathbf{B}_{tn}, \quad (n = 0, 1, \ldots), \tag{17}$$

where Δt is adopted time step.

The components of vector $\mathbf{B}_{tn}$ can be found from the following numerical differentiation formula:

$$(\mathrm{d}P_i/\mathrm{d}t)_n = [P_i(t_{n+1}) - P_i(t_n)]/\Delta t, \quad (i = 1, 2, \ldots, 55). \tag{18}$$

To calculate $P_i(t_{n+1})$ and $P_i(t_n)$ expressions (5) can be presented in the finite difference form with respect to time (see e.g. Korn and Korn, 1961; Wineman, 1980). Consider the length of time $0-t$. It can be divided into equal parts at intervals $\Delta t_m = t_{m+1} - t_m (m = 0, 1, 2, \ldots, n-1)$. The boundaries of the intervals are: $t_0 = 0$, t_1, $t_2, \ldots, t_{n-1}$, $t_n = t$. Hence, each integral in expressions (5) can be written as a sum of integrals over the intervals (t_m, t_{m+1}). Assume that the functions of nodal displacements $x_k(\tau)$ and, therefore, the integrands $f_{ij}(\tau) (j = 1, 2, \ldots, 5)$ are piecewise constants, i.e. their values remain unchanged within each interval $t_m \leqslant \tau \leqslant t_{m+1}$, however, they jump up at the interval boundaries. It is essential to note that 'left' approximation is employed in this case where the values of the smooth function under study and the step function substitute coincide on the left boundary of each interval. This permits using step-by-step computation beginning from the known initial value at $t_0 = 0$. Putting constant values of the integrands for each interval (t_m, t_{m+1}) outside the integral sign we get the approximate formulae for calculating P_i:

$$P_i^{(1)}(t_n) = \sum_{m=0}^{n-1} f_{i1}(t_m) \int_{t_m}^{t_{m+1}} R(t_n - \tau)\,\mathrm{d}\tau, \quad (i = 1, 2, \ldots, 39);$$

$$P_i^{(2)}(t_n) = \sum_{m=0}^{n-1} f_{i2}(t_m) \int_{t_m}^{t_{m+1}} R(t_n - \tau)\,\mathrm{d}\tau$$

$$-1,5 \sum_{j=3}^{5} \phi_{ij}(t_n) \sum_{m=0}^{n-1} f_{ij}(t_m) \int_{t_m}^{t_{m+1}} R(t_n - \tau)\,\mathrm{d}\tau, \quad (i = 40, 41, \ldots, 55). \tag{19}$$

Functions f_{i1}, f_{i2}, f_{ij}, ϕ_{ij} have the form (6).

Values $P_i(t_{n+1})$ and $P_i(t_n)$ which appear in (18), can be calculated from formulae (19) into which we substitute the values of nodal displacements x_k obtained at all the previous steps of problem solution with respect to time. In this case, to get the first approximation $P_i^{(2)}(t_{n+1})$ $(i = 40, 41, \ldots, 55)$ we use values of ϕ_{ij} calculated at displacement values from the previous step, i.e. $\phi_{ij}^{-}(t_{n+1}) = \phi_{ij}^{+}(t_n)$. Then, we should iterate to specify $P_i^{(2)}(t_{n+1})$ taking new values of ϕ_{ij} calculated at displacements obtained from solution of system (11) in a first approximation. The iteration is repeated as many times as necessary to achieve a certain specified degree of accuracy. In the region of convergence two or three iterations are usually sufficient. The iteration converges everywhere except for the vicinity of critical time, where the determinant of matrix **A** approaches zero value. Non-convergence of the iterative process here serves as an indicator suggesting loss of stability in the physical process.

The components of matrix $\mathbf{A}_n$ in (17) are calculated at the values of displacements given by vector $\mathbf{X}_n$. At the first step in solving the creep problem matrix $\mathbf{A}_0$ is calculated at displacement values given by the initial conditions of problem (12). If load $\bar{q}$ is taken as a driving parameter in solving the elastic-instantaneous problem (13), **A** matrices in the systems of equations for solving the elastic-instantaneous problem (15) and creep problem (11) must have the same form (10). Therefore, when the load reaches the given value, at the end of elastic solution and at the first step in solving the creep problem, the numerical values of **A** matrices will be equal to each other because they are calculated at the same values of displacements (12). We proceed from one problem to the other continuously by changing the driving parameter ($\bar{q}$ for t) and vector $\mathbf{B}_q$ for $\mathbf{B}_t$ in the right-hand side of the system of equations. Both problems can be solved within a unified calculation process. If the given load $\bar{q} < \bar{q}_{\mathrm{cr}}$, matrix $\mathbf{A}_0$ of the first step in solving creep problem is obviously a non-singular one. This allows us to do the first step in solving problem (17), followed by the next steps.

In integration of system (11) it may happen, in the course of time, that velocity of displacements $\mathrm{d}\mathbf{X}/\mathrm{d}t$ increases unboundedly, which is regarded, according to the adopted criterion, as the loss of stability in creep. With bounded damped creep of material this may occur when $\det \mathbf{A} \to 0$, i.e. when matrix **A** becomes singular. The components of displacements determined by solution (17) will be increasing unboundedly at that point. This instant will correspond to critical time t_{cr}. If $\bar{q} = \bar{q}_{\mathrm{cr}}$, it is obvious that $\det \mathbf{A}_0 = 0$ and $t_{\mathrm{cr}} = 0$.

The solution of the system of eqns (11) by Euler's method has the first order of error. If greater accuracy is required, a mulitstep Adams method can be applied in conjunction with multipoint approximation of derivatives $(\mathrm{d}P_i/\mathrm{d}t)_n$ (see e.g. Korn and Korn, 1961). The initial piece of the sought-for solution can be found through formulae (17) and (18). It should be noted, however, that the adopted accuracy of solution must be consistent with the accuracy of relaxation function determined experimentally. The rise of accuracy in solution cannot compensate an inaccuracy in input datas. Therefore with the requirement of greater accuracy for solution it is necessary to require that for relaxation function, in sense of the exact description by it of a material behaviour in time.

4. Results and discussion

Consider some results of calculations for hypar-shells. For linearly viscoelastic material of the shell the relaxation function, following the data in Arutynyan (1952), is defined as:

$$R(t-\tau) = k\gamma \exp[-\gamma(1+k)(t-\tau)], \tag{20}$$

where $k = 2.535$, $\gamma = 0.03\ \mathrm{d}^{-1}$. Poisson's ratio is $\nu = 0.2$. The curvature parameter for the hypar-shell is taken as $\bar{k}_{12} = 40$, 80, 160; the stiffness parameter for edge elements $\alpha_1 = 10$, 100. Consideration has been given to perfect and imperfect shells. The geometrical initial imperfections are taken as two elliptical dents in the upper quadrants of the shell. Their centres coincide with the centres of the upper quadrants. The axes of the dent ellipse are assumed to be: a along the diagonal of the shell, $0.7a$ across it. Along the axes of the ellipse the dent has the form of a single half-way sinusoid with amplitude w_{ia} in its centre. As shown by Ishakov (1993), such initial irregularities have the greatest effect on reduction of critical load on the elastic shell. To obtain the initial conditions (12) the elastic-instantaneous problem is solved by means of a step-by-step computation procedure with driving parameters $\bar{q}$ or $\bar{w}_0$ using Runge–Kutta's method. Euler's method is applied in solving the creep problem. Depending on the length of the time interval in question the step Δt is taken to be in the range of 0.75–10 days.

Figure 2 shows the characteristic graph of the time dependence of deflection at the shell centre $\bar{w}_0$ in creep as an example of a shell with parameter $\bar{k}_{12} = 40$. For comparison the perfect and imperfect shells have been calculated in creep with the same initial elastic deflections at the shell centre $\bar{w}_0$. The vertical dash–dot lines on the graph show critical time t_{cr} (days). As seen from the graph, the velocity of deflection increases unboundedly upon reaching the critical time, which is regarded as loss of stability in creep according to the adopted criterion. This point is characterized by snapping of the shell. Reduction of the load leads to a delay in the snapping moment occurrence. As the load decreases below as certain level, the calculation fails to detect the phenomenon of loss of stability within the observable time interval, since the increase in deflection is slowing down owing to damping of creep deformation. Thus, at the load $\bar{q} = 19.2$ the calculations with time interval of 600 days show no symptoms of approaching critical state. Similar curves $\bar{w}_0 - t$ are obtained for the other values of $\bar{k}_{12}$.

Figure 3 illustrated the effect of the edge elements compressibility in critical time. As seen from the graph, this effect is rather small.

Figure 4 shows the graph of load value $\bar{q}$ vs critical time t_{cr} for shells with different values of curvature parameter $\bar{k}_{12}$. First, consider perfect shells (continuous lines). The snapping moment for a given load is seen to depend on the shell curvature. Given the curvature, with increasing t_{cr} the function $\bar{q}(t_{cr})$ decreases monotonically, approaching asymptotically a certain level $\bar{q} = \bar{q}_{cr}^{\infty}$. The levels of load $\bar{q} = \bar{q}_{cr}^{\infty}$, shown in Fig. 4, can be taken as lower limits which $\bar{q}$ tends to reach

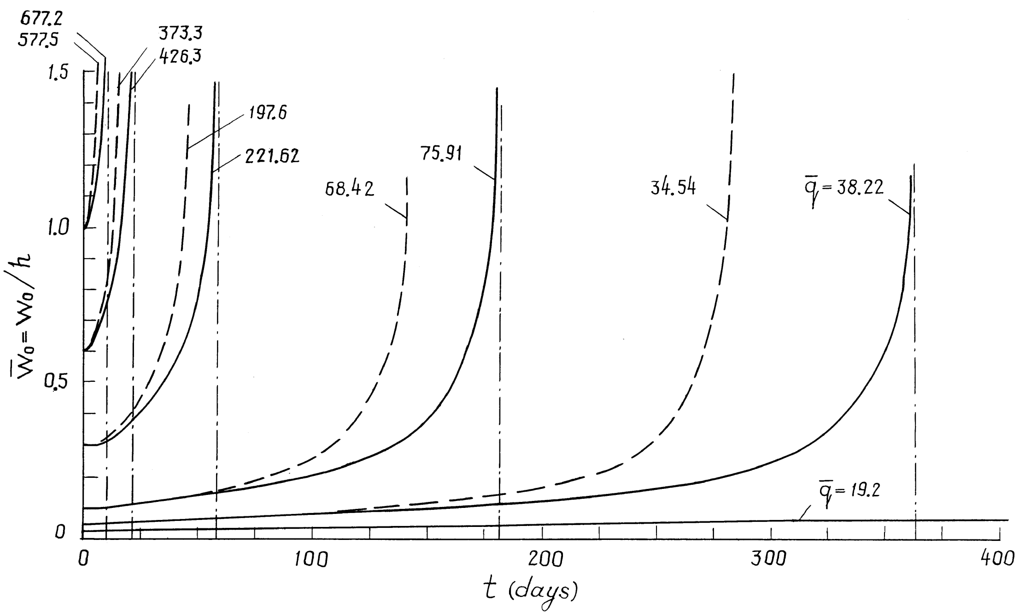

Fig. 2. Dependence of deflection at shell centre on the time in creep. Continuous lines are for the perfect shell, dashed lines are for the imperfect shell at $\bar{w}_{ia} = 0.25$. Shell parameters are: $\bar{k}_{12} = 40$, $\alpha_1 = 100$.

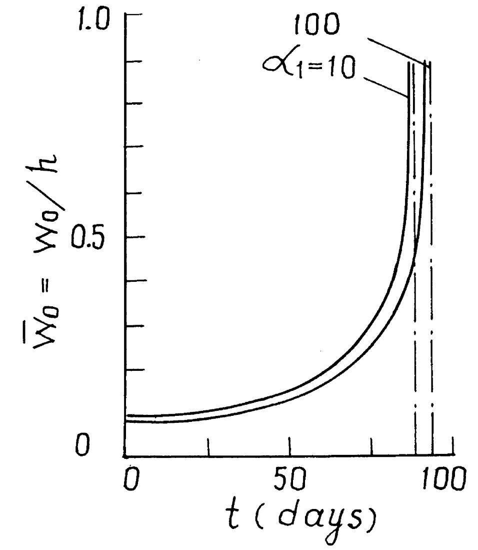

Fig. 3. Dependence of deflection at shell centre on the time in creep for different α_1, values. Shell parameters are: $\bar{k}_{12} = 80$, $\bar{w}_{ia} = 0$, $\bar{q} = 300$.

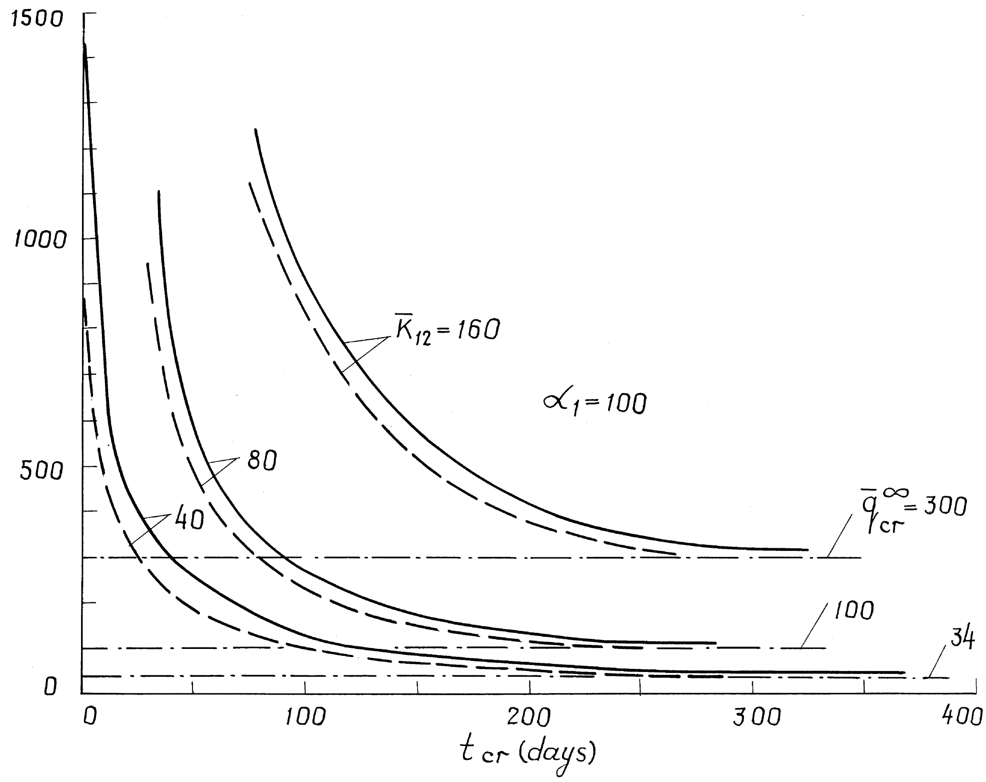

Fig. 4. 'Load-critical time' curves for hypar-shells. Continuous lines are for the perfect shells, dashed lines are for the imperfect shells at $\bar{w}_{ia} = 0.25$.

with $t_{cr} \to \infty$. Load $\bar{q}_{cr}^{\infty}$ can be considered as critical load for an outlying time moment. At load $\bar{q} \leqslant \bar{q}_{cr}^{\infty}$ the shell is stable in the half-infinite time interval $[0, \infty)$. At load $\bar{q} > \bar{q}_{cr}^{\infty}$ the stability must be estimated in the finite time interval $[0, t)$, during which the load is applied. The shell is considered stable in the time interval $[0, t)$ if $t_{cr} > t$.

For imperfect shells (dashed lines in Fig. 4) the character of the curves $\bar{q}(t_{cr})$ remains the same. With increasing t_{cr} the absolute gap in loading between the curves for perfect and imperfect shells narrows.

Figure 5 shows transformation of the perfect hypar-shell surface in the process of creep near the critical state. The lines represent equal increments in deflections from the point of loading till the one near the critical moment. Creep buckling dents develop mostly in the upper corner zones. This shape of buckling is similar to the primary buckling shape of identical elastic shell subjected to uniformly distributed load (Ishakov, 1993).

5. Conclusions

Creep buckling of flexible shallow hyperbolic paraboloid shells made of linearly viscoelastic material is accompanied by snapping when critical time is reached. As this point the velocity of

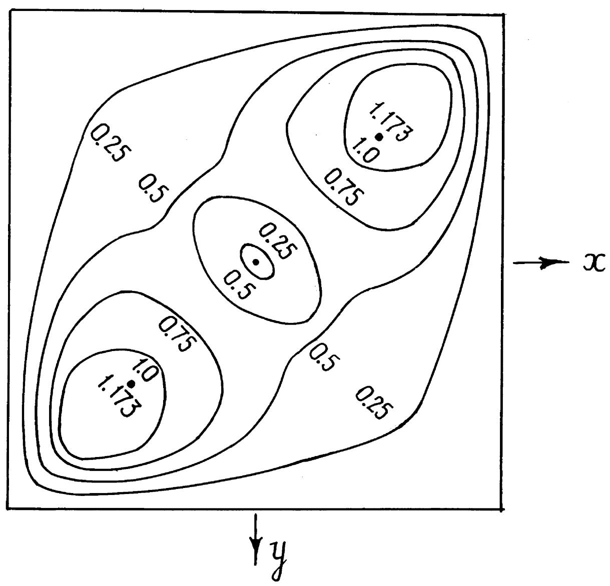

Fig. 5. Transformation on a perfect hypar-shell surface in creep. Shell parameters are: $\bar{k}_{12} = 160$, $\alpha_1 = 100$; $\bar{q} = 1164.65$; $t/t_{cr} = 0.86$.

deflections becomes infinitely large. With bounded damped creep of material for each curvature value there exists a load at which critical time increases unboundedly. It can be taken as critical load for an outlying time moment. If the given load is less than critical load for an outlying time moment, the shell is stable in the half-infinite time interval $[0, \infty)$, otherwise its stability should be estimated at the finite time interval $[0, t_{cr})$.

The approximate numerical method of solution suggested here may be applied in case of multistage loading of the shell. Switching from elastic to creep problems and vice versa at each stage can be performed by changing the driving parameters and vectors **B** in the right-hand side of the system of equations. The initial conditions for each stage in solving the problem are determined by the results of the last step in the previous solution.

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